The incompleteness of Mathematics and logic | By Marie Bonichon

An introduction to Kurt Gödel's incompleteness theorem

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It is often thought that mathematics is a science whose logical formalism must be perfect. However, the desire to axiomatize; to form new axioms--properties accepted without proof--as well as the advanced mathematics of the nineteenth century led the formal sciences to take measure of their intrinsic limitations.

Until the nineteenth century, axioms were mainly used for geometry. Mathematicians of the 1900s attempted to axiomatize arithmetics. On August 9, 1900, mathematicians such as Henri Poincare and Bertrand Russell met in Paris for the Second Congress of Mathematics. This is where David Hilbert, a German mathematician, provided 23 problems that aim to make mathematics evolve and to correct its imperfections. This new challenge has still not been met, since nine of these problems have still not been solved, and since it is now certain that we will never find the solutions to some of them. Indeed, in 1900, David Hilbert ignored the fact that mathematics cannot be a coherent and complete system; Kurt Gödel would be the one to later show that there is no solution to two of the 23 problems, due to systemic reasons--and not due to our ignorance. The incompleteness theorem was born, and the dream of complete mathematics died.

If we want to state this theorem in a simplified way, we say essentially this:

For any arithmetic theory, and even for any formal system in general, we admit the propositions of which the theory--or the system--can decide neither truth nor falsehood.

While it would be too ambitious to seek to expose the mathematical details of this theorem, we can still get a pretty clear sense of it through a famous logical paradox that is attributed to Bertrand Russell, which addresses the theory of sets:

1) What we call a normal set is a set that does not contain itself

2) What we call a non-normal set is a set that contains itself

3) Thus, what is the nature of the set of all normal sets?

4) If it is normal, it should not contain itself.

5) But it does contain normal sets, so it should be non-normal.

6) However if it is non-normal, it must contain itself, so it is the set of all non-normal sets, which doesn't make sense since it doesn't contain itself anymore and then must be normal, etc.

We see it here--propositions 5 and 6 are undecidable: the system of definition (meaning propositions 1 and 2) necessarily leads to the impossibility of determining the truth or falsity of propositions 5 and 6, which are, however, the immediate logical consequences. Russel proposes to include this theory by means of the following paradox, called the "paradox of the barber":

"The municipal council of a village stops an order requiring the village barber (male) to shave all the male inhabitants of the village who do not shave themselves, and only these males. The barber, who is a resident of the village, could not comply with this rule because:

- "If he shaves himself, he breaks the rule, because the barber can only shave men who do not shave themselves

- If he does not shave himself (whether he gets shaved by someone else or whether he keeps his beard), it is also wrong because he is in charge of shaving the men who do not shave themselves. This rule is therefore inapplicable. (...)"

For Kurt Gödel, it all began in 1906, in Vienna. Born into a Germanic family that owned a small textile factory, he had access to a good education during which he was noted for his scientific spirit and particular fascination with mathematics. He therefore continued his studies at the University of Vienna in physics and mathematics. He became a member of different groups of mathematicians, from which he moved away fairly quickly. In 1930, at age 24, he supported his thesis which reflects notably on the difference between mathematical and metamathematical propositions.

Metamathematical propositions are those which, in a calculation or demonstration, do not provide mathematical information. A proposition such as "let x be the variable" is metamathematical, while 1 +1 = 2 is a mathematical proposition.

His approach consists of combining the numbers for each metamathematical proposal so that the whole arithmetic system is coded. He then proceeds to demonstrate the existence of a number G such that G is not subject to any mathematical law--that is to say that G is the solution of any law. This number G let the mathematician state that "In any system finitely axiomatized, consistent and able of formalizing arithmetic, we can construct a proposal that can be neither proved nor refuted within this system. "

This basically means that: in all systems of codes that however still meet the mathematical laws, there exists a proposition that we must admit arbitrarily because we cannot show its legitimacy nor can we say that it is false: these are undecidable propositions. They are "undecidable" in the sense that they are only conjectures, and we do not know if they can be used as axioms or if they cannot be ignored, similarly to the propositions 5 and 6 of the paradox of normal sets. For example, the Goldbach conjecture is undecidable because it affirms that every even number is a sum of prime numbers (numbers with no whole divisors other than 1 and themselves). This seems fair if one notes the following relations: [8=3+5, 58=53+ ...]. However, this conjecture is still not proven, nor even disproved. Thus, mathematicians who use undecidable propositions are forced to make a choice in their reasoning; this means that they are forced to arbitrarily decide the value of truth or falsity of certain propositions. So, during a demonstration, if one mathematician uses the undecidable, he emits a hypothesis which allows him to obtain a result; nevertheless, the demonstration that he will have made before "deciding" remains genuine, but we can all of a sudden obtain several results for one single problem, because the beginning of the reasoning will be the same. Two mathematicians may decide to make two different choices. In this, Gödel showed that mathematics is not the complete and perfect science, and he dismantled a belief that goes back to Greek, or even Egyptian Antiquity.

Later Gödel would support his thesis by publishing a second theorem saying that in a coherent theory, what makes the theory consistent is undecidable relative to the theory ("If T is a coherent theory, any statement of T that affirms the coherency of T is an undecidable of T.”) Gödel definitively puts into question the definition of mathematics as a coherent and complete system.

Thus, thanks to the turn that mathematics may have experienced in the twentieth century and thanks to the emergence of paradoxes, Kurt Gödel was able to show that mathematics is a finite system which has its limits, like any other system. We are then entitled to ask the following question: does noting that a system is finite and incomplete, not refer to a higher category of logic, singular to even developing such a criteria of significance? In other words, what in the logic of our spirit allows us to raise the question of the complete or incomplete character of knowledge.

Does this mean, as Plato thought, that there is necessarily an infinite knowledge of which the finite knowledge of mathematics depends on later? Or, does this mean that there simply is no infinite knowledge? But, if any, from where does this questioning of the finite nature of the finite come from? Gödel's theorem shows that a finite system does not contain itself. In this case we must certainly admit that there is another moment of our thinking that is capable of producing such understanding. This moment of thought remains a logical moment, since it deals with determining the finite nature of formal logic. Therefore, there is always a logic of the spirit that precedes the only formal logic, a logic able of showing the finite character of the latter.